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Modified generalized Laguerre function Tau method for solving laminar viscous flow

The Blasius equation

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Abstract

Purpose – The purpose of this paper is to propose a Tau method for solving nonlinear Blasius equation which is a partial differential equation on a flat plate.

Design/methodology/approach – The operational matrices of derivative and product of modified generalized Laguerre functions are presented. These matrices together with the Tau method are then utilized to reduce the solution of the Blasius equation to the solution of a system of nonlinear equations.

Findings – The paper presents the comparison of this work with some well-known results and shows that the present solution is highly accurate.

Originality/value – This paper demonstrates solving of the nonlinear Blasius equation with an efficient method.

Keywords Flow, Laminar flow, Viscosity, Fluid dynamics, Differential equations

Paper type Research paper

1. Introduction

Recently, spectral methods have been successfully applied in the approximation of boundary value problems defined in unbounded domains. We can apply different approaches using spectral methods to solve problems in semi-infinite domains.

The first approach is using Laguerre polynomials and Laguerre functions (Guo and Shen, 2000; Maday *et al.*, 1985; Shen, 2000; Siyyam, 2001; Taghavi *et al.*, 2009). Guo and Shen (2000) suggested a Laguerre-Galerkin method for the Burgers equation and Benjamin-Bona-Mahony equation on a semi-infinite interval. It is shown that the Laguerre-Galerkin approximations are convergent on a semi-infinite interval with spectral accuracy. Shen (2000) proposed spectral methods using Laguerre functions and analyzed elliptic equations on regular unbounded domains. In Shen (2000) it is shown that spectral-Galerkin approximations based on Laguerre functions are stable and convergent with spectral accuracy in the Sobolev spaces. Maday *et al.* (1985) proposed a Laguerre type spectral method for solving partial differential equations. Siyyam (2001) applied two numerical methods for solving initial value problem using the Laguerre Tau method.

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The second approach is reformulating the original problem in semi-infinite domain to a singular problem in bounded domain by variable transformation and then using the Jacobi polynomials to approximate the resulting singular problem (Guo *et al.*, 2005).

The third approach is replacing the semi-infinite domain with interval [0, K] by choosing K, sufficiently large. This method is named as the domain truncation (Boyd, 2001).

The fourth approach of spectral method is based on rational orthogonal functions. Boyd (1987) defined a new spectral basis, named rational Chebyshev functions on the semi-infinite interval, by mapping to the Chebyshev polynomials. Guo *et al.* (2000) introduced a new set of rational Legendre functions which is mutually orthogonal in $L^2(0, +\infty)$. They applied a spectral scheme using the rational Legendre functions for solving the Korteweg-de Vries equation on the half line. Boyd *et al.* (2003) applied pseudospectral methods on a semi-infinite interval and compared rational Chebyshev, Laguerre, and mapped Fourier sine. Parand and Razzaghi (2004a, b, c) applied the spectral method to solve nonlinear ordinary differential equations on semi-infinite intervals. Their approach was based on a rational Tau method. They obtained the operational matrices of derivative and product of rational Chebyshev and Legendre functions and then applied these matrices together with the Tau method to reduce the solution of these problems to the solution of a system of algebraic equations. The Tau method was invented by Lanczos (1956) in 1938. In the current paper, our main aim is to employ the Tau method (Dehghan and Saadatmandi, 2006; Saadatmandi and Dehghan, 2008). The method is based on expanding the required approximate solution as the elements of a complete set of orthogonal functions. In the Tau method (Saadatmandi and Dehghan, 2007) unlike the Galerkin approximation, the expansion functions are not required to satisfy the boundary constraint individually (Canuto *et al.*, 1988).

The sections of this paper are organized as follows: in section 2, we describe the Blasius equation and transform it to a nonlinear ordinary differential equation and then explain some methods used previously to solve Blasius equation. In section 3, we describe the formulation of modified generalized Laguerre (MGL) functions required for our subsequent development. Section 4 summarizes the application of the MGL functions Tau method to the solution of nonlinear ordinary differential equation (ODE) for the Blasius equations. The operational matrices of the derivative and the product of MGL functions are derived in section 4. These matrices together with the Tau method are then utilized to evaluate the solution to the Blasius equation. As a result a set of nonlinear algebraic equations is formed, and the solution of the considered ODE is introduced. In this section, we use MGL functions to solve the Blasius equation and then compare our solutions with some well-known results, comparisons show that the present solutions are accurate. The conclusions are described in the final section, that is, Section 5.

2. Blasius equation

A great deal of interest has been focused on the steady flow of viscous incompressible fluids. Keulegen (1994) investigated the case of two parallel streams, where the upper stream was moving and the lower one was at rest. An approximate solution has been obtained for this model. Lock (1951) studied two cases, where the lower stream was at rest as well as when it was in motion. Potter (1951) extended the study to two fluids of different viscosities and densities, where both fluids were moving co-current with different velocities. The velocity distribution in the boundary layers is well addressed in Potter (1957). Recently, Abu-Sitta (1994) worked on a differential equation of mixing layer that occurs in Blasius equation. The existence of a solution for this model is successfully addressed and established in Abu-Sitta (1994) by using the technique of Weyl (1942). A broad class of analytical and numerical solutions in Abu-Sitta (1994), Ahmad and Al-Barakati (2009), Belhachmi *et al.* (2000), Datta (2003), Fang *et al.* (2006), Kuiken (1981a, b), Sweeney and Finaly (2007), Weyl (1942), Yu and Chen (1998) were used to handle this problem. It is well known that the Blasius equation is the most important of all boundary-layer equations in fluid mechanics. Many different but related equations have been derived for a multitude of fluid-mechanical situations, for instance, the Falkner-Skan equation (Belhachmi *et al.*, 2000). Taking into account the thermal radiation term in the energy equation, the governing equations of motion and heat transfer for the classical Blasius flat plate flow problem can be summarized by the following boundary value problem (Howarth, 1938; Raptis *et al.*, 2004):

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho c_P}\frac{\partial^2 T}{\partial y^2}-\frac{1}{\rho c_P}\frac{\partial q_r}{\partial y}. \tag{3}$$

The boundary conditions for the velocity field are:

$$u(x,0)=v(x,0)=0, \quad u(0,y)=U_\infty, \quad u(x,\infty)=U_\infty.$$

The thermal boundary conditions for the equation of energy equation (3) are:

$$T(x,0)=T_w, \quad T(0,y)=T_\infty, \quad T(x,\infty)=T_\infty.$$

Here u and v are (Howarth, 1938; Raptis *et al.*, 2004) the velocity components along the flow direction (x-direction) and normal to flow direction (y-direction), ν is the kinematic viscosity, k is the thermal conductivity, c_P is the specific heat of the fluid at a constant pressure, ρ is the density, q_r is the radiative heat flux, T is the temperature across the thermal boundary layer, T_w is a constant temperature of the wall, T_∞ is a constant temperature of ambient fluid ($T_\infty > T_w$) and U_∞ is a constant free stream velocity. It is assumed that the viscous dissipation is neglected, the physical properties of the fluid are constant, and the Boussinesq and boundary-layer approximations are valid. Bataller (2008), Magyari (2008), and Raptis *et al.* (2004) introduced a similarity variable η and a dimensionless stream function $f(\eta)$ as:

$$\eta=y\sqrt{\frac{U_\infty}{vx}}=\frac{y}{x}\sqrt{Re_x}, \tag{4}$$

$$\frac{u}{U_\infty}=f', \quad v=\frac{1}{2}\sqrt{\frac{U_\infty v}{x}}(\eta f'-f), \tag{5}$$

where a prime denotes differentiation with respect to η and Re_x is the local Reynolds number ($=U_\infty x/v$), and defined the nondimensional temperature $\theta(\eta)$ and the Prandtl number Pr as:

$$\theta(\eta) = \frac{T - T_w}{T_\infty - T_w}, \quad Pr = v\frac{\rho c_P}{k}, \tag{6}$$

to transform the nonlinear partial differential equations (1)-(3) to the following nonlinear ordinary differential equation.

$$f''' + \frac{1}{2}ff'' = 0, \tag{7}$$

$$\theta'' + \frac{Prk_0}{2}f\theta' = 0. \tag{8}$$

It is worth mentioning here that when $k_0 = 1$, the thermal radiation's effect is not considered. The transformed boundary conditions for the momentum equation (7) are:

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1. \tag{9}$$

The transformed thermal boundary conditions for the energy equation (8) are:

$$\theta(0) = 0, \quad \theta(\infty) = 1. \tag{10}$$

Realize that the momentum equation (7) is uncoupled from the energy equation (8) because the physical properties of the fluid are constant. Note that by replacing θ with f' and choosing k_0 so that $Prk_0 = 1$, the energy equation (8) and momentum equation (7) are equivalent. So by solving one of these equations we can obtain the other solution of the equation (Bataller, 2008; Magyari, 2008; Raptis *et al.*, 2004).

Here, we consider equation (7) which is the well-known Blasius equation which appears when studying a laminar boundary-layer problem for Newtonian fluids. Such a flow is usually called the boundary-layer flow, since the viscous effects are limited to a thin layer near the flat plate surface. Blasius (1908) solved the equation by using a series expansions method and found the following solution for the problem:

$$f(\eta) = \sum_{k=0}^{\infty}\left(-\frac{1}{2}^k\right)\frac{A_k\sigma^{k+1}}{(3k+2)!}\eta^{3k+2}, \tag{11}$$

where $A_0 = A_1 = 1$ and,

$$A_k = \sum_{r=0}^{k-1}\binom{3k-1}{3r}A_rA_{k-r-1}, \quad k \geq 2.$$

In equation (11), σ denotes the unknown $f''(0)$. In spite of the presence of $(3k + 2)!$ in the denominator, the above series converges only within a finite interval $[0, \eta_0]$ where $\eta_0 \approx 1.8894/\sigma$.

In the recent years, the study of the steady flow of viscous incompressible fluid has gained considerable interest because of its extensive engineering applications. Since the pioneering work of Howarth (1935) various aspects of the problem have been investigated by many authors. Squire (1959) used generalized Gauss-Laguerre quadrature to boundary-layer problems.

Liua and Chang (2008) developed a new numerical technique, they transformed the governing equation into a nonlinear second-order boundary value problem by a new transformation technique, and then solved it by the Lie-group shooting method. Wang (2004) employed the Adomian decomposition method (ADM) to solve numerically the famous Blasius equation. Hashim (2006) corrected the numerical solution of Wang and presented an improved numerical solution using the *ADM – Pade* Dehghan *et al.* (2009) approach. Wazwaz (2007) employed the variational iteration method (Dehghan and Shakeri, 2008) for a reliable treatment of two forms of the third-order nonlinear Blasius equation. He showed that the series solution is obtained without restrictions on the nonlinearity behavior. He combined the obtained series solution with the diagonal *Pade* approximants to handle the boundary condition at infinity for only one of these forms.

Wazwaz also used the modified decomposition method and *Pade* approximants for these equations. Liao (1999) used homotopy analysis method (HAM) to solve Blasius equation. Pahlavan and Boroujeni (2008) proposed a simple approach using HAM to obtain accurate analytical solution of viscous fluid flow over a flat plate. He (1998) approximated an analytical solution which was obtained with variational iteration method. The comparison with Howarth's numerical solution reveals that the proposed method is accurate. Tajvidi *et al.* (1999) used modified rational Legendre Tau method to solve Blasius equation. Lin (1999) obtained an approximate analytical solution of Blasius equation by the parameter iteration method.

Lastly Boyd (1999), calculated several numerical constants, such as the second derivative of Blasius equation at the origin and the two parameters of the linear asymptotic approximation to it, to at least 11 digits. Although the Blasius function is unbounded, Boyd nevertheless derive an expansion in rational Chebyshev functions *TLj* which converges exponentially fast with the truncation, and tabulate enough coefficients to compute it and its derivatives to about nine decimal places for all positive real x. The power series of f has a finite radius of convergence, but the Euler-accelerated expansion is apparently convergent for all real x. Boyd also showed that the singularities, which are first-order poles to lowest order, have an infinite series of cosine-of-a-logarithm corrections. Last, chart Boyd (1999) the behavior of f in the complex plane and conjecture that all singularities lie within three narrow sectors. Boyd (2008) also used this function to illustrate several important themes. He gave a list of interesting projects for undergraduates and another list of challenging issues for researchers and mathematicians.

3. Properties of the MGL functions

This section is devoted to the introduction of the basic notions and working tools concerning orthogonal MGL functions. More specifically, we presented some properties of the MGL functions.

3.1 The MGL functions

The Laguerre approximation has been widely used for numerical solutions of differential equations on infinite intervals. $L_n^\alpha(x)$ (generalized Laguerre polynomial) is the nth eigenfunction of the Sturm-Liouville problem (Bayin, 2006; Coulaud *et al.*, 1990; Guo and Shen, 2000):

$$x\frac{\mathrm{d}^2}{\mathrm{d}x^2}L_n^\alpha(x) + (\alpha+1-x)\frac{\mathrm{d}}{\mathrm{d}x}L_n^\alpha(x) + nL_n^\alpha(x) = 0,$$

$$x \in I = [0,\infty), \quad n = 0,1,2,\ldots.$$

The generalized Laguerre polynomials are defined with the following recurrence formula:

$$L_0^\alpha(x) = 1, \quad L_1^\alpha(x) = 1 + \alpha - x,$$

$$nL_n^\alpha(x) = (2n - 1 + \alpha - x)L_{n-1}^\alpha(x) - (n + \alpha - 1)L_{n-2}^\alpha(x),$$

these are orthogonal polynomials for the weight function $w_\alpha = x^a e^{-x}$. The generalized Laguerre polynomials satisfy the following relation:

$$\partial_x L_n^\alpha(x) = -\sum_{k=0}^{n-1} L_k^\alpha(x). \tag{12}$$

We define MGL functions ϕ_j as follows:

$$\phi_j(x) = \exp(-x/(2L))L_j^\alpha(x/L), \quad L > 0. \tag{13}$$

Where $\alpha = 0.5, 0.8, 1, 1.3, 1.5$, this system is an orthogonal basis (Gasper *et al.*, 1995; Taseli, 1996) with the weight function $w(x) = x/L$ and orthogonality property:

$$\langle \phi_n, \phi_m \rangle_{w_L} = \left(\frac{\Gamma(n+2)}{L^2 n!} \right) \delta_{nm},$$

where δ_{nm} is the Kronecker function. Boyd (1982, 1987, 2001) offered guidelines for optimizing the map parameter L where $L > 0$ is the scaling parameter. Numerical results depend smoothly on constant parameter L, and therefore are not very sensitive to L because the $dError/dL = 0$ at the minimum itself, so the error varies very slowly with L around the minimum. A little trial and error is usually sufficient to find a value that is nearly optimum. In general, there is no way to avoid a small amount of trial and error in choosing L when solving problems on an unbounded domain. Experience and the asymptotic approximations of Boyd (1982) can help, but some experimentation is always necessary as this author explains in his book Boyd (2001).

3.2 Function approximation

A function $f(x)$ defined over the interval $I = [0, \infty)$ can be expanded as:

$$f(x) = \sum_{i=0}^{\infty} a_i \phi_i(x), \tag{14}$$

where,

$$a_i = \frac{\langle f, \phi_i \rangle_w}{\langle \phi_i, \phi_i \rangle_w}. \tag{15}$$

If the infinite series in equation (14) is truncated with N terms, then it can be written as:

$$f(x) \simeq \sum_{i=0}^{N-1} a_i \phi_i(x) = A^T \phi(x), \tag{16}$$

with,

$$A=[a_0,a_1,a_2,\dots,a_{N-1}]^T, \tag{17}$$

$$\phi(x)=[\phi_0(x),\phi_1(x),\dots,\phi_{N-1}(x)]^T. \tag{18}$$

3.3 The derivative operational matrix

The derivative of the vector $\phi(x)$ defined in equation (13) can be expressed as:

$$\phi'(x)=D\phi(x), \tag{19}$$

where D is the $N\times N$ operational matrix for derivative. By taking the derivative of MGL functions we have the following relation:

$$\frac{d}{dx}\phi_x=-\frac{1}{2L}\exp(-x/(2L))L_n^{\alpha}(x/L)+\exp(-x/(2L))\frac{d}{dx}L_n^{\alpha}(x/L). \tag{20}$$

Using equations (12) and (20) the matrix D can be expressed. The matrix D is a lower triangular matrix with $-1/2L$ entries on the main diagonal and entries below the main diagonal are $-1/L$. For $N=5$ the matrix D is:

$$D=\frac{-1}{L}\begin{pmatrix}1/2&0&0&0&0\\1&1/2&0&0&0\\1&1&1/2&0&0\\1&1&1&1/2&0\\1&1&1&1&1/2\end{pmatrix} \tag{21}$$

3.4 The product operational matrix

The product of two MGL functions vectors defined in equations (13) can be expressed as:

$$\phi(x)\phi^T(x)A\simeq\tilde{A}\phi(x), \tag{22}$$

where $\tilde{A}$ is an $N\times N$ product operational matrix for the vector A. Using equation (22) and the orthogonal property, the elements $\tilde{A}_{ij}$, $(i,j=0,\dots,N-1)$ of the matrix $\tilde{A}$ can be calculated from (Tajvidi, 2008):

$$\tilde{A}_{i,j}=\left(L^2\frac{n!}{\Gamma(n+2)}\right)\sum_{k=0}^{N-1}a_k g_{ijk}, \tag{23}$$

where g_{ijk} is given by:

$$g_{ijk}=\int_0^{\infty}\phi_i(x)\phi_j(x)\phi_k(x)w(x)dx. \tag{24}$$

4. Solving the Blasius equation

In this section, we use MGL functions to solve the Blasius equation. For using MGL functions at first we multiply both sides of equation (7) by $e^{-x/2}$ so we have:

$$e^{-x/2}f''' + \frac{1}{2}e^{-x/2}ff'' = 0. \tag{25}$$

We now express all terms in equation (25) by MGL functions as:

$$e^{-x/2} = \sum_{i=0}^{N-1} e_i\phi_i(x) = E^T\phi(x), \tag{26}$$

where $E = [1,\ 0,\ 0, \ldots,\ 0]$. From equations (16), (19), and (22) we can deduce the following relations:

$$f^{(j)}(x) = \sum_{i=0}^{N-1} a_i\phi_i^{(j)}(x) = A^TD^j\phi(x), \quad j = 1, 2, 3, \tag{27}$$

where D^j is the jth power of the matrix D given in equation (21).

$$e^{-x/2}f'''(x) = A^TD^3\phi(x)\phi^T(x)E \simeq A^TD^3\tilde{E}\phi(x). \tag{28}$$

$$f(x)f''(x) \simeq A^T\phi(x)\phi^T(x)D^{2T}A. \tag{29}$$

If we set $F = D^{2T}A$, then equation (29) becomes:

$$f(x)f''(x) \simeq A^T\phi(x)\phi^T(x)F \simeq A^T\tilde{F}\phi(x), \tag{30}$$

$$f^{(3)}(x) = A^TD^3\phi(x). \tag{31}$$

Using equations (26)-(31) we get:

$$e^{-x/2}f(x)f''(x) = A^T\tilde{F}\phi(x)\phi^T(x)E = A^T\tilde{F}\tilde{E}\phi(x). \tag{32}$$

Using equations (28) and (32) the residual $Res(x)$ for equation (25) can be written as:

$$Res(x) = \left[A^TD^3\tilde{E} + \frac{1}{2}A^T\tilde{F}\tilde{E}\right]\phi(x). \tag{33}$$

As in a typical Tau method, we generate $N-2$ algebraic equations by applying:

$$\langle Res(x), \phi_i(x)\rangle = 0, \quad (i = 0, \ldots, N-3), \tag{34}$$

and from equation (9) we get:

$$y(0) = A^T\phi(0) = 0, \quad y'(0) = A^T D\phi(0) = 0. \tag{35}$$

Equation (34) with equation (35) generate a set of N nonlinear algebraic equations, that can be solved by Newton method for unknown coefficients a_j.

As mentioned before, the second derivative of $\alpha = f''(\eta)$ at zero plays an important role in the function. A high accurate numerical solution of Blasius equation has been provided by Howarth (1935), who obtained the initial slope $\alpha = f''(0) = 0.332057$. By homotopy perturbation method, He (1998, 2003) obtained the first iteration step led to 0.3095 with 6.8 percent accuracy (relative error), and the second iteration step yielded 0.3296 with 0.73 percent accuracy of the initial slope. Abbasbandy (2007) used the ADM and obtained $\alpha = f''(0) = 0.333329$ with 0.383 percent accuracy of the initial slope, also Tajvidi *et al.* (2008) calculated $\alpha = f''(0) = 0.33209$ with 0.009 percent accuracy. Howarth (1935) also obtained $f'(1) = 0.32979$ and $f(0) = 0.16557$. He (1998) obtained these with relative errors 16.68 and 6.32 percent, respectively. The approximations of $f''(0), f'(1), f(1)$ obtained by the method of the current paper show that our results are accurate. Table I gives odd coefficients in equation (16) with $\alpha = 1$ to evaluate f and its derivatives with an absolute error that is less than 10^{-6} for 21 terms.

In Table II, the resulting values of $f(1), f'(1)$ together with L and relative error which is less than 10^{-6} using the present method with $N = 7, 9, 11, 15, 19$ and $\alpha = 1$ are presented, respectively.

The approximations of the $\partial = f'(0)$ obtained by this method and their relative error which is defined as $(|\partial_{present\ method} - \partial_{Asaithambi}|/|\partial_{Asaithambi}|)$ with respect to the Asaithambi's (2005) results are listed in Table III.

Table IV, shows approximation of $f(\eta)$ for the present method with $\alpha = 0.5, 0.8, 1, 1.3, 1.5$, it seems that $\alpha = 1$ is the best choice. Tables V, VI, and VII show the numerical values of f, f', and f'' using the present method with $\alpha = 1$ and those of Rafael (2005) and Howarth (1935), respectively.

i	a_i	i	a_i
1	−1.123266292	13	−0.601874643E-4
3	0.983562976	15	0.42563546E-4
5	0.600965834	17	0.30067346E-4
7	−0.868982362E-2	19	−0.10012338E-5
9	0.138346295E-3	21	−0.5123563E-5
11	0.816453738E-3	23	0.2342642E-6

Table I. The first odd coefficients of equation (7)

N	L	$f(1)$	$error\ (\%)^f$	$f'(1)$	$error\ (\%)^{f'}$
7	0.79	0.16547	0.06	0.32853	0.38
9	0.81	0.16550	0.04	0.32959	0.06
11	0.82	0.16552	0.03	0.32966	0.03
15	0.85	0.16554	0.01	0.32970	0.02
19	0.88	0.16557	0.00	0.32979	0.00

Table II. The resulting values of $\partial = f(1), f'(1)$ for $\alpha = 1$ together with L and relative errors (%) using the present method

Figure 1 shows the resulting graph of Blasius (f, f') for $N = 6$, $\alpha = 1$, and $L = 1$. Figure 2 shows the convergence rate for f is remarkably fast; a_{21} is smaller than a_1 by roughly 10^{-6} (Boyd, 1999). Figure 3 shows logarithmic error of $\partial = f''(0)(|\partial_{present\ method} - \partial_{Asaithambi}|)$ vs N (number of terms) with respect to the Asaithambi's (2005) results for $N = 21$ and $\alpha = 1$.

Table III. Approximations of $\partial = f''(0)$

		Present method		Liao (1999)		Pahlavan and Boroujeni (2008)	
N	L	σ	Relative error (%)	Order	∂	Order	∂
7	0.79	0.332064634	0.002	5	0.28098	4	0.327531
9	0.82	0.332059633	0.0007	10	0.32992	6	0.330855
11	0.89	0.332058638	0.0003	15	0.33164	8	0.331503
15	0.98	0.332057736	0.49E-5	20	0.33198	10	0.331807
21	1.00	0.332057524	0.15E-7				

Notes: Approximations obtained by the present method and its relative error with Asaithambi's result (2005) and the methods used by Liao (1999) and Pahlavan and Boroujeni (2008)

Table IV. Approximation of $f(\eta)$ for the present method with $\alpha = 0.5, 0.8, 1, 1.3, 1.5$

η	$\alpha = 0.5$	$\alpha = 0.8$	$\alpha = 1$	$\alpha = 1.3$	$\alpha = 1.5$
0.00	0.00000	0.00000	0.00000	0.00000	0.00000
1.00	0.16558	0.16558	0.16557	0.16557	0.16559
2.00	0.65009	0.65007	0.65003	0.65003	0.65002
3.00	1.39685	1.39684	1.39682	1.39679	1.39676
4.00	2.30582	2.30581	2.30576	2.30574	2.30563
5.00	3.28331	3.28331	3.28329	3.28326	3.28326
6.00	4.27965	4.27966	4.27964	4.27963	4.27962
7.00	5.27927	5.27929	5.27926	5.27926	5.27924
8.00	6.27930	6.27931	6.27923	6.27923	6.27921
9.006	7.27931	7.27932	7.27923	7.27922	7.27921

Note: $n = 21$ terms

Table V. Approximation of $f(\eta)$ for the present method with $\alpha = 1$, solutions of Rafael (2005) and Howarth (1935)

η	Present method	Rafael (2005)	Solutions of Howarth (1935)
0.00	0.0000000	0.00000	0.00000
1.00	0.1655731	0.16557	0.16557
2.00	0.6500351	0.65003	0.65003
3.00	1.3968254	1.39682	1.39682
4.00	2.3057619	2.30576	2.30576
5.00	3.2832913	3.28330	3.28329
6.00	4.2796473	4.27965	4.27964
7.00	5.2792619	5.27927	5.27926
8.00	6.2792353	6.27923	6.27923
9.00	7.2792383	7.27925	7.27923

Note: $n = 21$ terms

Table VI.
Approximation of $f'(\eta)$ for the present method, solutions of Rafael (2005) and Howarth (1935)

η	Present method	Rafael (2005)	Solutions of Howarth (1935)
0.00	0.0000000	0.00000	0.00000
1.00	0.3297956	0.32978	0.32979
2.00	0.6297737	0.62977	0.62977
3.00	0.8460586	0.84605	0.84605
4.00	0.9555253	0.95552	0.95552
5.00	0.9915583	0.99155	0.99155
6.00	0.9989882	0.99898	0.99898
7.00	0.9999272	0.99993	0.99992
8.00	1.0000000	1.00000	1.00000
9.00	1.0000000	1.00000	1.00000

Note: $n = 21$ terms

Table VII.
Approximation of $f''(\eta)$ for the present method, solutions of Rafael (2005) and Howarth (1935)

η	Present method	Rafael (2005)	Solutions of Howarth (1935)
0.00	0.3320542	0.33206	0.33206
1.00	0.3230174	0.32301	0.32301
2.00	0.2667514	0.26675	0.26675
3.00	0.1613615	0.16136	0.16136
4.00	0.0642426	0.06423	0.06424
5.00	0.0159142	0.01591	0.01591
6.00	0.0024067	0.00240	0.00240
7.00	0.0002228	0.00022	0.00022
8.00	0.0000100	0.00001	0.00001
9.00	0.0000000	0.00000	0.00000

Note: $n = 21$ terms

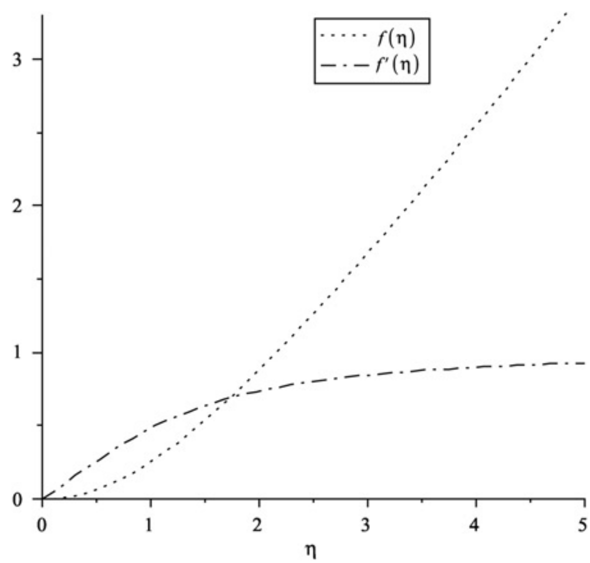

Figure 1.
Graph of the approximations of $f(\eta)$ (dotted line) and $f'(\eta)$ (dashed-dotted line) for Blasius equation obtained by the present method

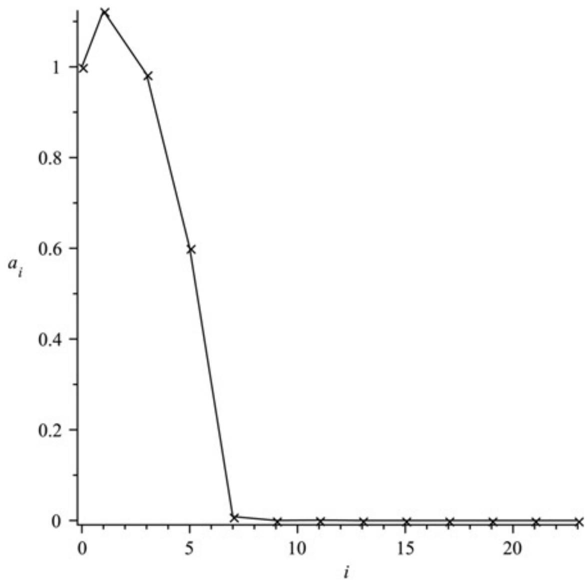

Figure 2.
Absolute values of coefficients of the MGL function a_j of the Blasius function

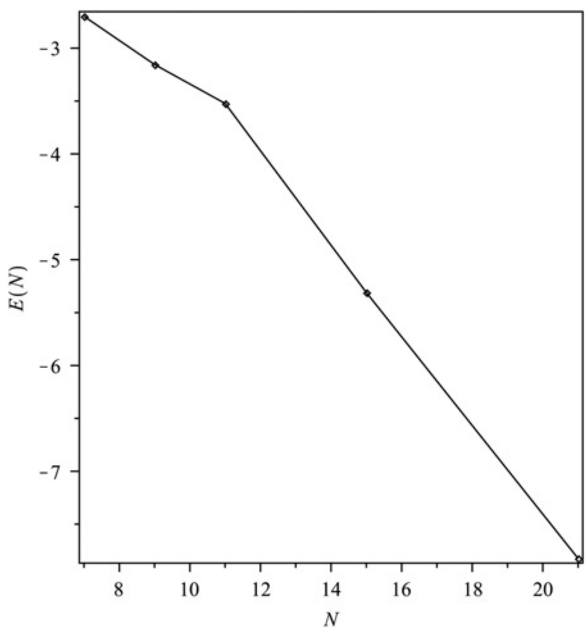

Figure 3.
Logarithmic error of $\partial = f''(0)$ ($|\partial_{present\ method} - \partial_{Asaithamu}$) vs N (number of terms) with respect to Asaithambi's (2005) results

5. Conclusions

In this paper, we considered the Blasius equation, which is a laminar viscous flow over a semi-infinite flat plate. Blasius equation occurs in the study of laminar boundary-layer problem for Newtonian fluids. The difficulty in this type of equations, due to the existence of its boundary condition in infinity, is treated here. In the Blasius equation, the second derivative at zero is an important point of the function, so we have computed $f''(0)$ and have compared it with other results. The fundamental goal of this paper has been to construct an approximation to the solution of nonlinear Blasius equation in a semi-infinite interval. A set of orthogonal functions proposed to provide an effective but simple way to improve the convergence of the solution by Tau method. Note that if the results of the Blasius equation are more accurate, the energy equation (8) can be solved with a high accuracy. The results are found to be in excellent agreement with the exact solution. All of our computations verify that the proposed procedure offers an effective tool for solving this nonlinear problem in fluid mechanics. It has been shown that the present work with small N provides accurate solutions for the Blasius equations.

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